Closing Tues: 2.8
Closing next Thurs: 3.1-2
Closing next Fri: 3.3 (last before Exam 1)

2.8 Derivative Function (continued)

Sometimes "slope of tangent" doesn't make sense.

Def'n: We say a function, $y = f(x)$, is **<u>differentiable</u>** at $x = a$ if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \text{'a number'}.$$

Otherwise it is not differentiable at a.

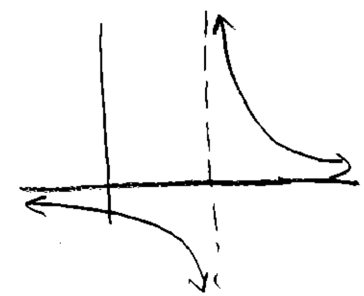
To be differentiable:

1. It must be defined at x = a.
2. It must be continuous at x = a.
3. Same "slope" from "both sides".

Examples (try to sketch these):

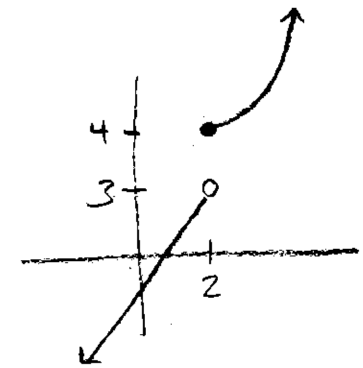
1. $f(x) = \frac{1}{x-3}$
2. $g(x) = \begin{cases} 2x - 1 & \text{, if } x < 2; \\ x^2 & \text{, if } x \geq 2. \end{cases}$
3. $k(x) = |x|$
4. $j(x) = x^{1/3}$

$\boxed{1}$



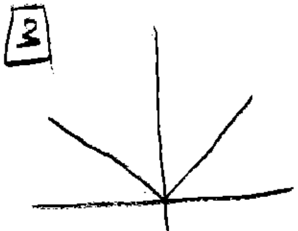
NOT DEFINED AT x = 3
⇒ NOT CONT AT x = 3
⇒ NOT DIFF. AT x = 3

$\boxed{2}$



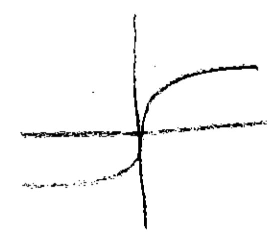
NOT CONT. AT x = 2
⇒ NOT DIFF. AT x = 2

$\boxed{3}$



slopes don't match at x = 0
NOT DIFF AT x = 0

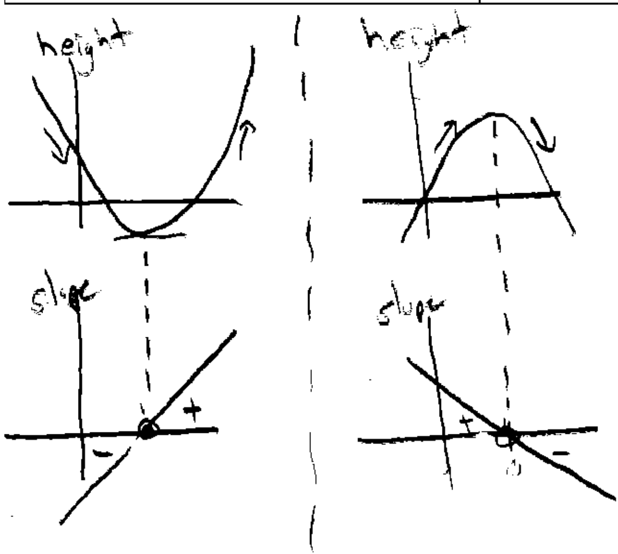
$\boxed{4}$



vertical tangent at x = 0
("infinite slope")
NOT. DIFF. AT x = 0

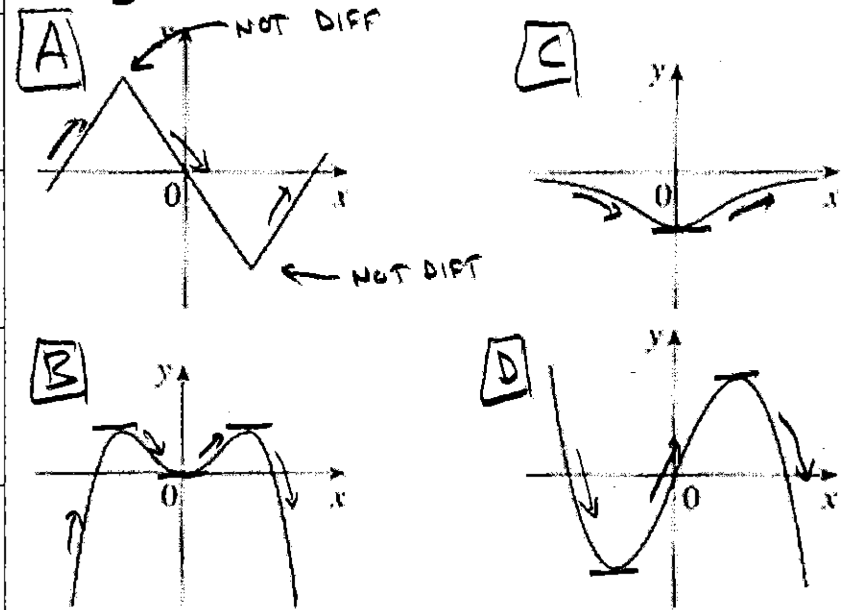
Matching Derivative Graphs

$y = f(x)$	$y = f'(x)$
horizontal tangent	zero (crosses x-axis)
increasing (uphill)	positive (above *x*-axis)
decreasing (downhill)	negative (below *x*-axis)
not differentiable	undefined

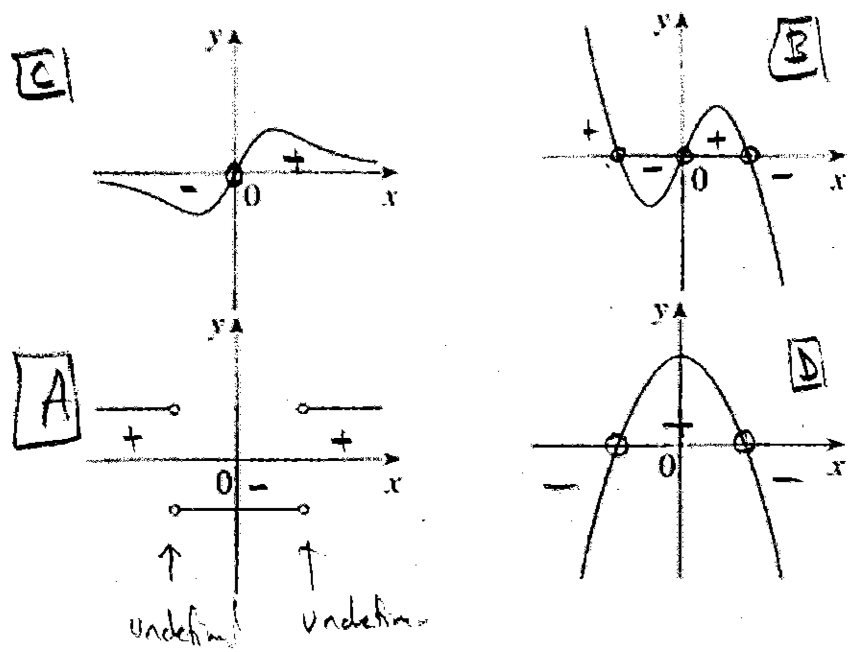


From homework: Match the graphs

Original Functions



Derivatives



Derivative Notation

Last time, we found

if $\quad f(x) = 2x^2 - 3x,$

then $\quad f'(x) = 4x - 3.$

Other ways to write this include:

$$y' = 4x - 3$$

$$\frac{dy}{dx} = 4x - 3$$

$$\frac{d}{dx}(2x^2 - 3x) = 4x - 3.$$

2nd Derivatives

Later we will also discuss:

$$f''(x) = y'' = \frac{d(dy/dx)}{dx} = \frac{d^2y}{dx^2}$$

Example:

if $\quad y = f(x) = 2x^2 - 3x,$

then $\quad y' = f'(x) = 4x - 3$

and $\quad y'' = f''(x) = 4$

which can also be written as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x - 3) = 4$$

3.1/3.2 Intro to Derivative Rules

Some Basic Limit Laws:

1. $\dfrac{d}{dx}(c) = 0$

Ex) $f(x)=10$, $f'(x)=0$

$\frac{d}{dx}(7)=0 \qquad \frac{d}{dx}(2^4)=0$

$\frac{d}{dx}(\ln(2))=0 \qquad \frac{d}{dx}(e^3)=0$

2. $\dfrac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

Ex) $\frac{d}{dx}(4x+3) = \frac{d}{dx}(4x)+\frac{d}{dx}(3)$

$= 4 + 0$

3. $\dfrac{d}{dx}\big(cf(x)\big) = cf'(x)$

Ex) $\frac{d}{dx}(10x) = 10\frac{d}{dx}(x)$

$= 10 \cdot 1 = 10$

"Proof"

1. Constant Rule: For $f(x)=c$,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

2. Sum rule:

$$\lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

3. Constant coefficient rule:

$$\lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$4.\ \frac{d}{dx}(x^n) = nx^{n-1}$$

$f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

$f(x) = x^3$

$(x+h)(x+h)(x+h)$

$(x^2+2xh+h^2)(x+h)$

$x^3+3x^2h+3xh^2+h^3$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

"Proof"

4. Power Function Rule: For $f(x) = x^n$,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + h^2(\ldots) - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + h^2(\ldots)}{h}$$

$$= \lim_{h \to 0} nx^{n-1} + h(\ldots) = nx^{n-1}$$

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

PASCAL'S TRIANGLE

$f(x) = x^4$

$$\lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{x^4}}{h}$$

$$= 4x^3$$

Algebra Skills Test

Rewrite each term in the form: $a\,x^b$, and find the derivative:

1. $y = 2 + 7\sqrt{x^3} + \dfrac{13}{2x^6}$

$y = 2 + 7x^{3/2} + \frac{13}{2}x^{-6}$

$y' = 0 + 7\,\frac{3}{2}x^{1/2} + \frac{13}{2}(-6x^{-7})$

$\boxed{y' = \frac{21}{2}\sqrt{x} - 36x^{-7}}$

2. $y = \dfrac{32 \cdot 15\,x^4}{16 \cdot 5\,x^6} + \dfrac{x^7 x^2}{4(x^2)^3} + \dfrac{3}{x}$

$y = 6x^{-2} + \frac{1}{4}\,\frac{x^9}{x^6} + 3x^{-1}$

$y = 6x^{-2} + \frac{1}{4}x^3 + 3x^{-1}$

$y' = 6(-2x^{-3}) + \frac{1}{4}(3x^2) + 3(-x^{-2})$

$\boxed{y' = -12x^{-3} + \frac{3}{4}x^2 - 3x^{-2}}$

3. $y = 17\sqrt[5]{x^3} + 3(2x^5)$

$y = 17x^{3/5} + 6x^5$

$y' = 17(\frac{3}{5}x^{-2/5}) + 6(5x^4)$

$\boxed{y' = \frac{51}{5}x^{-2/5} + 30x^4}$

4. $y = x^2(3x^3 - 4x)$

$y = 3x^5 - 4x^3$

$\boxed{y' = 15x^4 - 12x^2}$

$$5. \frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

WE DEFINE e TO BE THE NUMBER SUCH THAT

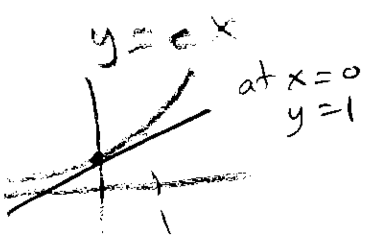
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

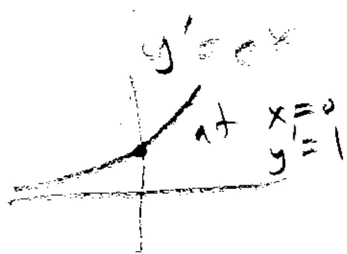
BY EXPERIMENTATION THIS NUMBER IS $e \approx 2.71828182\ldots$

FURTHERMORE,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a)$$

NOTE $\ln(e) = 1$





"Proof"

5. Exponential Function Rule:

For $f(x) = a^x$,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Note: $\ln(a) = \lim_{h \to 0} \frac{a^h - 1}{h}$

ASIDE:

NUMERICALLY WE CAN EXPLORE THIS LIMIT.

$$\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693147$$

$$\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.098612$$

$$\lim_{h \to 0} \frac{2.7182^h - 1}{h} \approx 0.999970$$

Find the derivative

1. $y = 5x + 3x^3 + 7e^x$

$y' = 5(1) + 3(3x^2) + 7e^x$

$\boxed{y' = 5 + 9x^2 + 7e^x}$

2. $y = \dfrac{4(2)^x}{3} - 11 + \dfrac{8\sqrt[3]{x^2}}{5}$

$y = \frac{4}{3}(2)^x - 11 + \frac{8}{5}x^{2/3}$

$y' = \frac{4}{3}(2)^x \ln(2) - 0 + \frac{8}{5}\frac{2}{3}x^{-1/3}$

$\boxed{y' = \frac{4}{3}(2)^x \ln(2) + \frac{16}{15}x^{-1/3}}$

اجباري

$$6.\ \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

$$7.\ \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

"Proof"

6. Product Rule:

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Find the derivatives:

1. $y = \underbrace{x^4}_{f(x)}\underbrace{e^x}_{g(x)}$

F S' - F'S

$\quad f \quad g' \quad - \quad f' \quad g$

$$\boxed{\begin{aligned} y' &= x^4 e^x - 4x^3 e^x \\ y' &= (x^4 - 4x^3)e^x \end{aligned}}$$

2. $y = \dfrac{2x^3}{x^2+4}$

$f \leftarrow N$

$g \leftarrow D$

$\dfrac{DN' - ND'}{D^2}$

$$y' = \frac{(x^2+4)(6x) - 2x^3(2x)}{(x^2+4)^2}$$

$$\boxed{y' = \frac{6x^3 + 24x - 4x^4}{(x^2+4)^2}}$$

3. $y = (\sqrt{x} + 4x)3^x - \dfrac{14}{x^5}$

$$y = (\underbrace{\sqrt{x}}_{x^{1/2}} + 4x)3^x - 14x^{-5}$$

F S

$$\boxed{y' = (\sqrt{x} + 4x)\,3^x \ln(3) + (\tfrac{1}{2}x^{-1/2} + 4x)\,3^x - 14(-5x^{-6})}$$

4. $y = 6(x+3)^2 + \dfrac{e^x}{x^3}$

$$y = 6(x^2+6x+9) + e^x/x^3$$

$$y = 6x^2 + 36x + 54 + \frac{e^x \leftarrow N}{x^3 \leftarrow D}$$

$$y' = 12x + 36 + 0 + \frac{x^3(e^x) - e^x(3x^2)}{x^6}$$

$$y' = 12x + 36 + \frac{\cancel{x^2}e^x(x-3)}{x^{\cancel{6}\,4}}$$